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Spectral distribution of drums with fractal perimeters: The Weyl-Berry-Lapidus conjecture

Yutaka Hobiki, Kousuke Yakubo, and Tsuneyoshi Nakayama
Department of Applied Physics, Hokkaido University, Sapporo 060, Japan
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The Weyl-Berry-Lapidus conjecture on vibrational spectra of drums with fractal perimeters (fractal drums) is examined in terms of large-scale numerical simulations. The integrated densities of states of fractal drums with the perimeter of the Koch curve are computed by employing a powerful numerical method. It is confirmed that the Weyl-Berry-Lapidus conjecture holds in the frequency regime higher than a crossover frequency ω_c related to the length scale characterizing the fractal boundary.

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Many objects in nature possess irregular and fractal geometries [1,2]. Dynamical properties of such structures have attracted much attention in recent years [3]. Among these, drums with fractal perimeters, so called fractal drums, provide various physical implications, e.g., scattering of waves by fractal surfaces, water waves in lakes (seiches), and oscillations of the earth [4]. The asymptotic properties of the integrated density of states (IDOS) of fractal drums in the high-frequency regime have been discussed from a mathematical point of view [5,6]. Lapidus [6] has conjectured that the IDOS, $I_{\rm fix}(\omega)$, of fractal drums with fixed boundary conditions should have the following frequency dependence in the high-frequency limit:

$$I_{\rm fix}(\omega)=\frac{S}{4\pi}\,\omega^2-B_f\omega^{D_f}, \qquad (1)$$

where D_f is the fractal dimension of the perimeter, S is the area of the drum, and B_f is a positive constant depending on a shape of the drum. This expression is called the Weyl-Berry-Lapidus (WBL) conjecture. The first term of Eq. (1) is the IDOS of fractal drums with *free* boundary in the high-frequency limit. The second term is a correction due to the vanishing degree of freedom by imposing fixed boundary conditions at a fractal perimeter. The degree of freedom is estimated by assuming a linear dispersion relation for boundary modes excited in the fractal drum with free boundary conditions.

The WBL conjecture has been numerically studied by Sapoval *et al.* [7,8] via the calculation of the IDOS of fractal drums with the Koch-curve perimeter (Koch drum) with the fractal dimension $D_f=1.5$. They have used a mapping relation between the Helmholtz equation and the diffusion equation for computing the IDOS, and calculated the eigenfunctions and the eigenfrequencies from diffusion processes. The frequencies in their calculation, however, are not high enough. In addition, the generation of the Koch drum reflecting the fractality of the boundary was not sufficient.

The purpose of the present paper is to discuss the validity of the WBL conjecture on the spectral distribution of fractal drums via large-scale simulations. We confirm, by employing the Koch drum in higher generation and a powerful numerical method to treat very large systems, that the WBL conjecture is valid in the higher-frequency regime than a crossover frequency ω_c related to the length scale characterizing the fractal boundary.

The WBL conjecture assumes that a drum is a continuum medium. We transform the Helmholtz equation for the continuum drum into discretized equations of motion with a grid spacing a using the central difference approximation. We replace the first term of Eq. (1) by the IDOS of a drum with free boundary, and regard the following relation as the discretized version of the WBL conjecture,

$$I_{\rm fix}(\omega)=I_{\rm free}(\omega)-B_f\omega^{D_f}. \qquad (2)$$

The validity of this relation has been checked by calculating the IDOS of a square drum. The result is in fairly good agreement with Eq. (2), leading to $D_f=1$. In order to confirm the validity of Eq. (2) for fractal drums, we introduce the difference $C(\omega)$ between $I_{\rm free}(\omega)$ and $I_{\rm fix}(\omega)$,

$$C(\omega)=I_{\rm free}(\omega)-I_{\rm fix}(\omega). \qquad (3)$$

The Koch drums we treat are illustrated in Fig. 1. The vibrating area is taken to be constant on various generations of the Koch drum, but the lengths of perimeters increase with increasing generation. We examine, at first, the Koch drum in the third-generation ($\nu=3$), which seems to reflect sufficiently the fractality of the boundary shape. The length of the segment l [see Fig. 1(c)] at the boundary must be taken much larger than the grid spacing a. Here the value of a is chosen to be $l=29a$. Therefore, the system size L at $\nu=3$ becomes $L=3\,074a$, and the number of grid points is $N_s=3\,474\,433$, excluding the number of perimeter sites because of the fixed boundary condition. The number of sites on perimeter is $N_p=59\,392$.

Our fractal drum consists of the grid points of N_s sites with mass m $(=1)$ and linear springs, with force constant K_{ij}, connecting nearest-neighbor sites. The equations for elastic vibrations are expressed by

$$m\ddot{\psi}_i=\sum_j K_{ij}\psi_j, \qquad (4)$$

where ψ_i is the displacement of the ith site. The spring constant for the nearest-neighbor interaction is taken to be $K_{ij}=1$. We apply the forced oscillator method (FOM) for the calculation of the IDOS [9–11]. This method enables us to

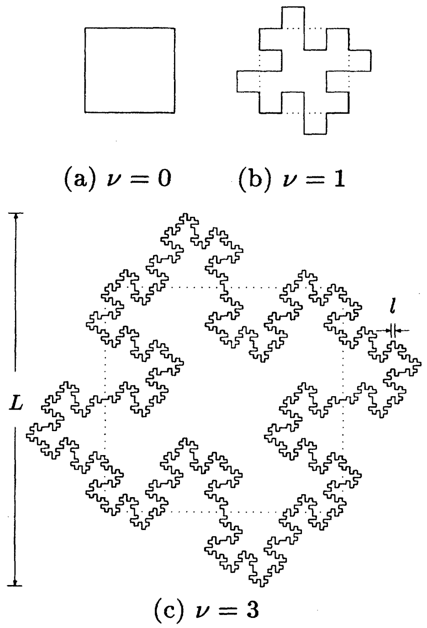

FIG. 1. Illustrations of the Koch drums in various generations. (a) The initiator of the Koch drum. (b) The Koch drum in the first generation, $\nu=1$. (c) The Koch drum in $\nu=3$. The dotted lines shown in (b) and (c) indicate the initiator. The fractal dimension of the Koch drum is $D_f = \ln 8/\ln 4 = 1.5$. Note that the segment l is the length scale characterizing the fractal boundary.

calculate quite accurately the densities of states (DOS) of very large systems. In this numerical technique, the DOS, $D(\omega)$, of fractal drums are obtained from the averaged total energy of the system $\langle E(t,\omega)\rangle$ under applying the periodic external force $F_i = F_0 \cos\varphi_i \cos\Omega t$, where F_0 is a constant, φ_i is random quality on the ith site, and Ω is a frequency of the external force. The average $\langle\ \rangle$ is taken over a number of sets $\{\varphi_i\}$. The DOS is related to the total energy $\langle E(t,\omega)\rangle$ through the relation

$$D(\omega) = \frac{8\langle E(t,\omega)\rangle}{\pi t F_0^2 N_s}. \tag{5}$$

The IDOS, $I_{\rm fix}(\omega)$, is obtained by integrating $D(\omega)$. We can obtain the IDOS of the Koch drum with a great number of grid points without calculating the eigenfrequencies. The advantages of the FOM lie also on being easily vectorized and parallelized for implementation on an array-processing modern supercomputer. This is due to the fact that the time-consuming part in computations is to solve the equation of motion and the program is easily optimized.

Figure 2 shows our numerical results on the frequency dependence of the calculated IDOS. Open circles and crosses denote the IDOS when taking free and fixed boundary conditions, respectively. Filled circles are the correction term $C(\omega)$ obtained using Eq. (3). The result clearly shows that the correction term $C(\omega)$ is proportional to ω^{D_f} with $D_f = 1.5$ in the higher-frequency regime than a crossover frequency ω_c which indicates the applicability of the WBL conjecture.

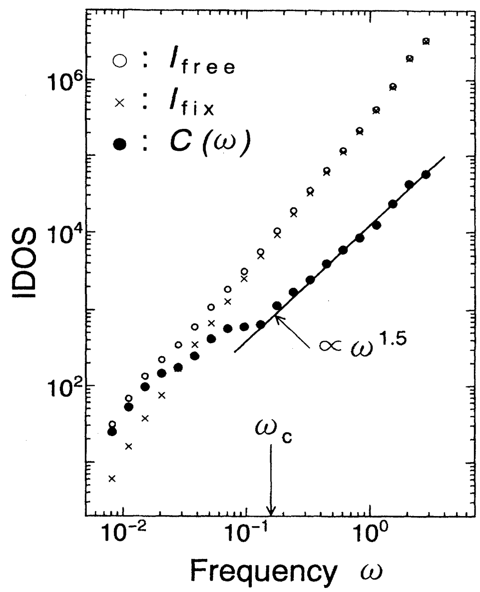

FIG. 2. The IDOS, $I_{\rm free}(\omega)$ and $I_{\rm fix}(\omega)$, for the Koch drum at $\nu=3$ as a function of frequency. Open circles and crosses indicate the IDOS for drums with free and fixed boundary conditions, respectively. Filled circles indicate the correction term $C(\omega)$ which are proportional to $\omega^{1.5}(=\omega^{D_f})$ in the higher frequency regime than a crossover frequency ω_c. In the lower frequency regime than ω_c, $C(\omega)$ does not follow the power law dependence ω^{D_f}.

The physical interpretation of ω_c can be given by comparing the wavelength λ of vibrational excitations and the length of the segment l at the boundary as follows: The WBL conjecture is valid in the high-frequency limit or short wavelength limit ($\lambda \to 0$). We assume that vibrations of fractal drums with free boundary can be separated into the inside and the boundary part for the case $\lambda/l \ll 1$. Provided that λ/l is larger than unity, vibrational excitations are strongly scattered at the fractal boundary, and one cannot separate the modes into the inside and the boundary parts or, even if possible, the assumption of the linear dispersion relation of the boundary modes is not appropriate. This implies that the WBL conjecture fails even in the very high-frequency region if the conditions $\lambda/l \gg 1$ holds. We claim that the crossover frequency ω_c is related to the excitation mode with the wavelength $\lambda \approx 2l$ (see Fig. 3).

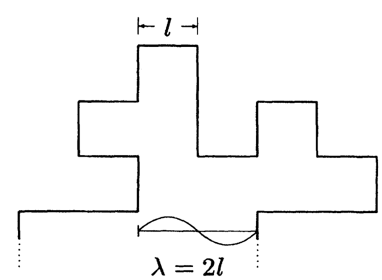

FIG. 3. A schematic illustration for the crossover frequency ω_c corresponding to the wavelength $\lambda = 2l$.

Let us estimate the value of the crossover frequency ω_c for our Koch drum. Since $l=29a$, ω_c should become 0.153 by using the linear dispersion relation $\omega=2\pi/\lambda\sqrt{K/m}$ for the boundary modes in the drum with free boundary conditions, where we use the system of unit $m=K=1$. This value agrees with a crossover frequency ω_c obtained in Fig. 2. We see that the WBL conjecture is valid when the wavelength λ of vibrational excitations of fractal drums is shorter than the length scale l characterizing the fractal boundary.

In the low-frequency regime $\omega \ll \omega_c$ (or $\lambda \gg l$), the WBL conjecture does not hold due to the following possibilities: (a) The separation of modes into the inside and the boundary part is not acceptable, and (b) the dispersion becomes anomalous (see Ref. [3]) due to fractality, namely, the assumption of the linear dispersion is not appropriate. In case (a), the correction term $C(\omega)$ for the region $\omega \ll \omega_c$ would not show the power law dependence for frequency. For case (b), the correction term $C(\omega)$ should be proportional to ω^α, where the exponent α takes a different value from the fractal dimension D_f of the perimeter.

In order to clarify this point, we have computed the IDOS of the Koch drum in the fifth generation ($\nu=5$). The length of the fractal segment l is taken to be $2a$, so that the system size at $\nu=5$ is $L=3\,412a$, and the degree of freedom is $N_s=4\,325\,377$ (excluding the perimeter sites: $N_p=262\,144$). The calculated results are shown in Fig. 4 as a function of frequency ω. The symbols in Fig. 4 are the same with those given in Fig. 2. A crossover frequency ω_c becomes 2 assuming a linear dispersion. Note that, due to rather high crossover frequency $\omega_c \approx 2$, the frequency region calculated in Fig. 4 almost covers the condition $\lambda \gg l$. We see that the correction term $C(\omega)$ does not follow the power law dependence, indicating that relevant modes are inseparable. We are planning to compute the IDOS of drums with different shapes in order to confirm our results. The preliminary results support our conclusions reported in this paper.

In summary, we have computed the IDOS of fractal drums with the Koch-curve ($D_f=1.5$) perimeters in generations $\nu=3$ and 5. The WBL conjecture has been confirmed numerically. The results show that (i) the criterion on the validity of the WBL conjecture has become clear, namely the conjecture holds when the wavelength λ of vibrational excitations are shorter than the length scale l characterizing the fractality of the perimeter, and it fails when the wavelength λ becomes longer than the length scale l, and (ii) the WBL conjecture is applicable not only under the high-frequency limit but also under the condition $\lambda/l \to 0$.

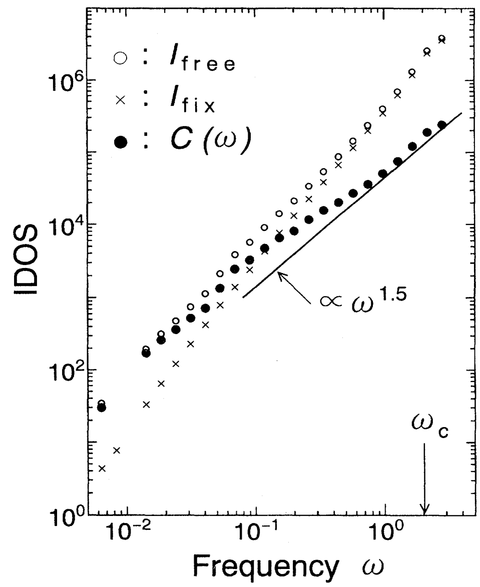

FIG. 4. The IDOS, $I_{\mathrm{free}}(\omega)$ and $I_{\mathrm{fix}}(\omega)$, for the Koch drum at $\nu=5$ as a function of frequency ω. The symbols are the same with those in Fig. 2. The crossover frequency ω_c becomes 2. The correction term $C(\omega)$ does not follow the power law dependence below $\omega_c \approx 2$.

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